Integration By Substitution

Let f and g be functions that satisfy the conditions of the Chain Rule for the composite function y = $f(g(x))$. If F is an antiderivative of f, then

$$\int f(g(x))g'(x)dx = F(g(x)) + c$$

Let u = $g(x)$. Then $du = g'(x)dx$ and $\int f(g(x))g'(x)dx = \int f(u)du.$

Examples:

1. $\int -2xe^{-x^2}dx = ?$ Notice, this integral has the form $\int f'(x)e^{f(x)}dx$

Let $u = -x^2$. Then $du = -2x\, dx$ and

$\int -2xe^{-x^2}$	
$\int e^{-x^2}(-2xdx)$	Rewrite
$\int e^u du$	Substitute
$e^u + c$	Integrate
$e^{-x^2} + c$	Substitutebackin for "u"

2. $\int -xe^{-x^2}dx = ?$

Let $u = -x^2$. Then $du = -2x\, dx$, get $(-x\, dx)$ by itself so $\frac{du}{2} = -x\,dx$, and

$\int -xe^{-x^2}dx$	
$\int e^{-x^2}(-xdx)$	Rewrite
$\int e^u \frac{du}{2}$	Substitute
$\frac{1}{2}\int e^u du$	Pull the constatnt to the front
$\frac{1}{2}e^u + c$	Integrate
$\frac{1}{2}e^{-x^2} + c$	Substitute back in for "u"

3. $\int \cos^2 x (\sin x)\, dx = ?$

Let $u = \cos x$. Then $du = -\sin x\, dx$, so $-du = \sin x\, dx$, and

$\int \cos^2 x(\sin x)dx$	Rewrite
$\int u^2(-du)$	Substitute
$-\int u^2 du$	Bring negative to front
$-\frac{1}{3}u^3 + c$	Integrate
$-\frac{1}{3}(\cos x)^3 + c$	Substitute back in for "u"
$-\frac{1}{3}\cos^3 x + c$	

4. $\int \frac{e^{2x}dx}{\sqrt{e^x - 1}} = ?$

Let $u = e^x - 1$. Then $du = e^x dx$ and $(u + 1) = e^x$.

Since $e^{2x} = e^x * e^x = (e^x)^2$, then $(u + 1)^2 = e^{2x}$, and

$$\int \frac{e^{2x}dx}{\sqrt{e^x - 1}}$$

$\int \frac{e^x(e^x dx)}{\sqrt{e^x - 1}}$ Rewrite

$\int \frac{e^x du}{\sqrt{u}}$ Substitute

There is an "x" left so we solve for e^x using $e^x - 1 = u \Rightarrow e^x = (u+1)$

$\int \frac{(u+1)du}{\sqrt{u}}$	Substitute again
$\int u^{-1/2}(u+1)du$	Bring $\sqrt{u}$ up to the numerator
$\int u^{1/2} + u^{-1/2}du$	Distribute $u^{-1/2}$
$\frac{2u^{3/2}}{3} - 2u^{1/2} + c$	Integrate
$\frac{2(e^x - 1)^{3/2}}{3} - 2(e^x - 1)^{1/2} + c$	Substitute back in for "u"

5. $\int x\sqrt{2x+1}\,dx = ?$

$= \left(e^x - 1\right)^{\frac{1}{2}}\left(\frac{2}{3}\left(e^x - 1\right) + 2\right) + c$

$= \frac{2}{3}\sqrt{e^x - 1}\left(e^x + 2\right) + c$

Let $u = 2x + 1$, then $du = 2dx$, $dx = \frac{du}{2}$, and $x = \frac{u-1}{2}$

$\int x\sqrt{2x+1}\,dx$

$\int x\sqrt{u}\,\frac{du}{2}$ Substitute

An "x" is left over so slove for "x" using u = 2x + 1; $x = \frac{u - 1}{2}$

$\int \left(\frac{u-1}{2}\right)\sqrt{u}\,\frac{du}{2}$ Substitute again

$\frac{1}{4}\int (u-1)u^{\frac{1}{2}}du$ Bring constant to the front

$\frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}du$ Distribute $u^{\frac{1}{2}}$

$\frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + c$ Integrate

$\frac{1}{4}\left(\frac{2}{5}(2x+1)^{\frac{5}{2}} - \frac{2}{3}(2x+1)^{\frac{3}{2}}\right) + c$ Substitute back in for "u"

$\frac{(2x+1)^{\frac{5}{2}}}{10} - \frac{(2x+1)^{\frac{3}{2}}}{6} + c$ Mulitply $\frac{1}{4}$ through

6. $\int \frac{dy}{y^2 - 2y + 5} =$

$$\int \frac{dy}{y^2 - 2y + 5} = \int \frac{dy}{\left(y^2 - 2y + 1\right) + 4} = \int \frac{dy}{(y-1)^2 + 4}$$

Let $u = y - 1$. Then $du = dy$, $\int \frac{dy}{(y-1)^2 + 4} = \int \frac{du}{u^2 + 2^2}$, and

we can now use the integration formula to complete the integration:

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + c,\quad \text{where } a \text{ is a constant.}$$

$$\int \frac{du}{2^2+u^2} = \frac{1}{2}\arctan\left(\frac{u}{2}\right) + c = \frac{1}{2}\arctan\left(\frac{y-1}{2}\right) + c$$

Otherwise, let $u^2 = 4x^2$, then $u = 2x$ and $du = 2dx$. Then,

$$\int \frac{dy}{y^2-2y+5} = \int \frac{du}{u^2+2^2} = \int \frac{2dx}{4x^2+4} = \frac{2}{4}\int \frac{dx}{(x^2+1)} = \frac{1}{2}\arctan x + c$$

$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + c = \frac{1}{2}\arctan\left(\frac{y-1}{2}\right) + c$$

Bibliography: Larson, E. Roland & Hostetler, P. Robert. Calculus with analytic geometry, 3rd edition. D.C. Heath and Company, Lexington, 1986.