DERIVATIVES I
Chain Rule, Power Rule

Chain Rule
If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f[g(x)]$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{or } equivalently, \qquad \frac{dy}{dx} = y' = f'[g(x)]g'(x).$$

In applying the Chain Rule, think of the composite function $f \circ g$ as having an inside and an outside part:

$$y = f\,\underbrace{[g(x)]}_{\substack{u=g(x)\\ inside}} = \underbrace{f(u)}_{outside}$$

General Power Rule a special case of the Chain Rule.

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx} \text{ or } equivalently, \quad \frac{d}{dx}[u^n] = nu^{n-1}u'$$

Examples: Find the derivative of each function given below.

1. $y = (x^2 - 3x + 5)^{25}$

   Let $u = x^2 - 3x + 5.$ Then $\frac{du}{dx} = 2x - 3,\quad y = u^{25},\quad \frac{dy}{du} = 25u^{24}$, and
   $\frac{dy}{dx} = (25u^{24})(2x-3) = 25(x^2 - 3x + 5)^{24}(2x-3)$

2. $\sqrt{3x^2 - 2x + 3}$

   Let $u = 3x^2 - 2x + 3$, Then $\frac{du}{dx} = 6x - 2,\quad y = \sqrt{u} = u^{1/2},\quad \frac{dy}{du} = \frac{1}{2}u^{-1/2}$
   $\frac{dy}{dx} = \frac{1}{2}u^{-1/2}(6x-2) = \frac{1}{2}(3x^2 - 2x + 3)^{-1/2} \cdot (6x-2) = \frac{6x-2}{2\sqrt{3x^2 - 2x + 3}}$

3. $y = 5\sqrt[3]{x^2 + \sqrt{x^3}}$

Rewrite the expression : $y = 5\left(x^2 + x^{3/2}\right)^{1/3}$

Let $u = x^2 + x^{3/2}$. Then, $\frac{du}{dx} = 2x + \frac{3}{2}x^{1/2}$, $y = 5u^{1/3}$, $\frac{dy}{du} = \frac{5}{3}u^{-2/3}$, and

$$\frac{dy}{dx} = \left(\frac{5}{3}u^{-2/3}\right)\left(2x + \frac{3}{2}x^{1/2}\right) = \frac{5}{3}\left(x^2 + x^{3/2}\right)^{-2/3}\left(2x + \frac{3}{2}x^{\frac{1}{2}}\right)$$

4. $y = e^{3x^2 - 5x + 5}$

Let $u = 3x^2 - 5x + 5$. Then, $\frac{du}{dx} = 6x - 5$, $y = e^u$, $\frac{dy}{du} = e^u$, and

$$\frac{dy}{dx} = e^u \cdot (6x - 5) = e^{3x^2 - 5x + 5} \cdot (6x - 5)$$

5. $\ln\left(2x^2 + 3x\right)$

Let $u = 2x^2 + 3x$, Then $\frac{du}{dx} = 4x + 3$, $y = \ln u$, $\frac{dy}{du} = \frac{1}{u}$, *and*

$$\frac{dy}{dx} = \frac{1}{u} \cdot \left(4x + 3\right) = \frac{4x + 3}{2x^2 + 3x}$$

6. $y = \ln(\cos x^2)$.

Let $u = \cos x^2$ Then $\frac{du}{dx} = (-\sin x^2)(2x)$, $y = \ln u$, $\frac{dy}{du} = \frac{1}{u}$, and

$$\frac{dy}{dx} = \left(\frac{1}{u}\right)(-\sin x^2)(2x) = -\frac{2x \sin x^2}{\cos x^2} = -2x \tan x^2$$

This example illustrates how the Chain Rule is to be used **all the way through**.