THE SIMPLEX METHOD

An algebraic technique that applies to any number of variables and enables us to solve larger linear programming problems is called the **simplex method.**

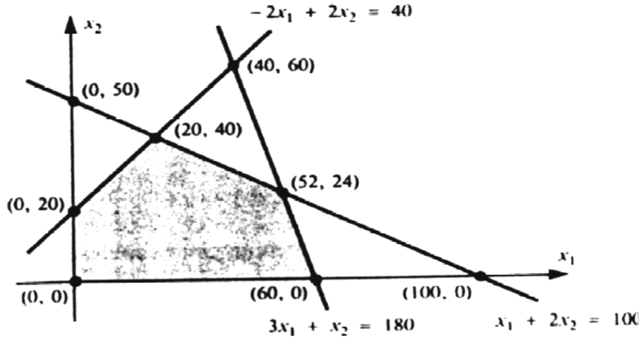
A linear programming problem may look like:

maximize: $z = 4x_1 + 12x_2$, subject to
$3x_1 + x_2 \leq 180$
$x_1 + 2x_2 \leq 100$
$-2x_1 + 2x_2 \leq 40$
$x_1 \geq 0,\ x_2 \geq 0$

To solve this problem graphically, graph the feasible region formed by the lines:

$3x_1 + x_2 = 180$
$x_1 + 2x_2 = 100$
$2x_1 + 2x_2 = 40$
$x_1 = 0,\ x_2 = 0$

this region is shown below:



Examine the boundary points: (0, 0), (60, 0), (52, 24), (20, 40), and (0, 20) and substitute these values for $\mathbf{x_1}$ and substitute $\mathbf{x_2}$ into the equation: $z = 4x_1 + 12x_2$

Point	**The value of *z***
(0, 0)	4(0) + 12(0) = 0 + 0 = 0
(60, 0)	4(60) + 12(0) = 240 + 0 = 240
(52, 24)	4(52) + 12(24) = 208 + 288 = 496
(20, 40)	4(20) + 12(40) = 80 + 480 = 560
(0, 20)	4(0) + 12(20) = 0 + 240 = 240

The maximum value of **z** is 560 corresponding to the point (20, 40).
The answer is $\mathbf{x_1 = 20, x_2 = 40}$, and **z = 560**.

Solve the same system using the simplex method:

Step one: Convert the constraints to linear equations by introducing additional variables called **slack variables.** The first constraint, $3x_1 + x_2 \le 180$, is true for pairs of numbers such as $x_1 = 5$ and $x_2 = 17$ because $3(5)+17 \le 180$. Also note that $3x_1 + x_2 + 130 = 180$ when $x_1 = 10$ and $x_2 = 20$. In this case, the **slack variable** for the first constraint (denoted by $\mathbf{s_1}$) is 130. The objective function, $z = 4x_1 + 12x_2$, can be rewritten as: $-4x_1 - 12x_2 + z = 0$

With the addition of slack variables, the constraints become:

$$\begin{array}{llllll}
3x_1 + x_2 + s_1 & & & & = 180 \\
x_1 + 2x_2 & + s_2 & & & = 100 \\
-2x_1 + 2x_2 & & + s_3 & & = 40 \\
-4x_1 - 12x_2 & & & + z & = 0 \\
\multicolumn{5}{l}{x_1 \ge 0, x_2 \ge 0, s_1 \ge 0, s_2 \ge 0, s_3 \ge 0}
\end{array}$$

Now form the augmented matrix of this system:

$$\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
3 & 1 & 1 & 0 & 0 & 0 & 180 \\
1 & 2 & 0 & 1 & 0 & 0 & 100 \\
-2 & 2 & 0 & 0 & 1 & 0 & 40 \\
-4 & -12 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Step 2: Find the **pivot element.** Look at the last row and pick out the **most negative entry** (-12). This selects the **pivot column** (1, 2, 2). See above.

Now find the **pivot row**. Divide each constant (180, 100, 40) by the corresponding entries in the pivot column:

$$\begin{array}{r|cccccc|c|l}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & & \\
 & 3 & 1 & 1 & 0 & 0 & 0 & 180 & 180/1 = 180 \\
 & 1 & 2 & 0 & 1 & 0 & 0 & 100 & 100/2 = 50 \\
\text{pivot row} \rightarrow & -2 & 2 & 0 & 0 & 1 & 0 & 40 & 40/2 = 20 \\
 & -4 & -12 & 0 & 0 & 0 & 1 & 0 & \\
 & & \uparrow & & & & & & \\
 & & \textit{pivot column} & & & & & &
\end{array}$$

Select the **smallest ratio** (to get the pivot row):

1. Ignore all negative ratios.
2. If a zero ratio occurs and if the zero was obtained by dividing a zero by a negative number, then ignore the zero. If the smallest ratio was obtained by dividing a zero by a positive number, use that row for the pivot row.

The smallest ratio is 20 which determines the pivot row. The **pivot element** is (2) in the second column, third row.

Step 3: Pivoting. As in matrices, make the pivot element "1" and the rest of the pivot column "0's." Then, divide the third row by 2 to get:

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 3 & 1 & 1 & 0 & 0 & 0 & 180 \\ 1 & 2 & 0 & 1 & 0 & 0 & 100 \\ -1 & 1 & 0 & 0 & 1/2 & 0 & 20 \\ -4 & -12 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Put zeros above and below the pivot element:

$$\begin{array}{r} (-1)R3 + R1 \rightarrow R1 \\ (-2)R3 + R2 \rightarrow R2 \\ (12)R3 + R4 \rightarrow R4 \end{array} \left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 4 & 0 & 1 & 0 & -1/2 & 0 & 160 \\ 3 & 0 & 0 & 1 & -1 & 0 & 60 \\ -1 & 1 & 0 & 0 & 1/2 & 0 & 20 \\ -16 & 0 & 0 & 0 & 6 & 1 & 240 \end{array}\right]$$

Step 4: Ask "Is 'z' maximized?"

If the last row contains any negative numbers, **z** is **NOT** maximized, repeat steps 2, 3, and 4.

To find another **pivot element**, look at the last row and pick out the most negative entry (-16). This selects the **pivot column** (4, 3, -1) above.

To get the **pivot row**, divide each constant (160, 60, 20) by the corresponding entries in the pivot column:

$$\begin{array}{r} \\ \\ \text{pivot row } \rightarrow \\ \\ \\ \\ \end{array} \begin{array}{c} \left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 4 & 0 & 1 & 0 & -1/2 & 0 & 160 \\ 3 & 0 & 0 & 1 & -1 & 0 & 60 \\ -1 & 1 & 0 & 0 & 1/2 & 0 & 20 \\ -16 & 0 & 0 & 0 & 6 & 1 & 240 \end{array}\right] \\ \begin{array}{l} \uparrow \\ \text{pivot column} \end{array} \end{array} \begin{array}{l} \\ 160/4 = 40 \\ 60/3 = 20 \\ 20/-1 = -20 \\ \\ \\ \end{array}$$

The smallest ratio is 20 which determines the pivot row. The **pivot element** is (3) in the first column second row.

Make the pivot element "1" and the rest of the pivot column "0's." Then, divide the second row by 3 to get:

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 4 & 0 & 1 & 0 & -1/2 & 0 & 160 \\ 1 & 0 & 0 & 1/3 & -1/3 & 0 & 20 \\ -1 & 1 & 0 & 0 & 1/2 & 0 & 20 \\ -16 & 0 & 0 & 0 & 6 & 1 & 240 \end{array}\right]$$

Put zeros above and below the pivot element:

$$\begin{array}{r} (-4)R2 + R1 \rightarrow R1 \\ (1)R2 + R3 \rightarrow R3 \\ (16)R2 + R4 \rightarrow R4 \end{array} \quad \begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 1 & -4/3 & 5/6 & 0 & 80 \\ 1 & 0 & 0 & 1/3 & -1/3 & 0 & 20 \\ 0 & 1 & 0 & 1/3 & 1/6 & 0 & 40 \\ 0 & 0 & 0 & 16/3 & 2/3 & 1 & 560 \end{array}$$

Is z maximized?

The last row contains no negative numbers; therefore, **z is maximized**.

What is the final answer?

Notice that the first, third and last columns' are **unit columns,** so the basic variables are x_1, x_2, s_1, and z. The feasible solution is obtained by setting s_2 and $s_3 = 0$ and solving for the others:
$x_1=20$, $x_2=40$, $s_1=80$, $s_2=0$, $s_3=0$, $z=560$

This is the same answer obtained in the first page.

Now try to solve a minimization problem:

minimize: $z = 3x_1 + 5x_2 + 2x_3$, subject to
$6x_1 + 9x_2 + 12x_3 \leq 672$
$x_1 - x_2 + 2x_3 = 92$
$5x_1 + 10x_2 + 10x_3 \geq 480$
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Summary of the Simplex Method for Minimization Problems:

1. Maximize w = -z.

2. (≥ constraint) For EACH constraint of the form: $a_1x_1 + a_2x_2 +$ __ $+ a_nx_n \geq b$, multiply the inequality by -1 to obtain: $-a_1x_1 - a_2x_2 -$ __ $- a_nx_n \leq -b$.

3. Replace (= constraint) of the form: $a_1x_1 + a_2x_2 +$ __ $+ a_nx_n = b$, with two: $a_1x_1 + a_2x_2 +$ __ $+ a_nx_n \leq b$ and $a_1x_1 + a_2x_2 +$ __ $+ a_nx_n \geq b$. The later is written as: $-a_1x_1 - a_2x_2 -$__$- a_nx_n \leq -b$

4. Form the initial table.

5. If NO negative entry appears in the last column of the initial table, proceed to Phase II.

6. (Phase I) There is at least one negative entry in the last column:
   a. The pivot row, the row containing the most negative entry in the last column.
   b. The pivot column, select the most negative entry in the pivot row (left to right).
   c. Pivoting on the pivot column.

7. Repeat step 6 till you have no negative entries in the last column.

8. (Phase II) Use the standard simplex (maximization procedure) to obtain a solution.

Using the simplex method, this problem is modified as the following:

maximize:

$w = -z = -3x_1 - 5x_2 - 2x_3$, subject to
$6x_1 + 9x_2 + 12x_3 \le 672$
* $x_1 - x_2 + 2x_3 \le 92$
* $x_1 - x_2 + 2x_3 \ge 92$
* $-5x_1 - 10x_2 - 10x_3 \le -480$
$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

Note the changes in the marked rows (*).

Phase I. The initial table is

$$\left[\begin{array}{cccccccc|c} 6 & 9 & 12 & 1 & 0 & 0 & 0 & 0 & 672 \\ 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 92 \\ -1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & -92 \\ -5 & -10 & -10 & 0 & 0 & 0 & 1 & 0 & -480 \\ 3 & 5 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Pivot on -10 in Row 4, Column 2 to obtain

$$\begin{array}{r} R4 \div -10 \to R4 \\ (-9)R4 + R1 \to R1 \\ R4 + R2 \to R2 \\ (-1)R4 + R3 \to R3 \\ (-5)R4 + R5 \to R5 \end{array} \left[\begin{array}{cccccccc|c} 3/2 & 0 & 3 & 1 & 0 & 0 & 9/10 & 0 & 240 \\ 3/2 & 0 & 3 & 0 & 1 & 0 & -1/10 & 0 & 140 \\ -3/2 & 0 & -3 & 0 & 0 & 1 & 1/10 & 0 & -140 \\ 1/2 & 1 & 1 & 0 & 0 & 0 & -1/10 & 0 & 48 \\ 1/2 & 0 & -3 & 0 & 0 & 0 & 1/2 & 1 & -240 \end{array}\right]$$

Pivot on -3 in Row 3 to obtain

$$\begin{array}{r} R3 \div -3 \to R3 \\ (-3)R3 + R1 \to R1 \\ (-3)R3 + R2 \to R2 \\ (-1)R3 + R4 \to R4 \\ R3 + R5 \to R5 \end{array} \left[\begin{array}{cccccccc|c} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 0 & -1/3 & -1/30 & 0 & 140/3 \\ 0 & 1 & 0 & 0 & 0 & 1/3 & -1/15 & 0 & 4/3 \\ 2 & 0 & 0 & 0 & 0 & -1 & 2/5 & 1 & -100 \end{array}\right]$$

Phase II. Pivot on the 1 in Row 2, Column 6 to obtain

$$\begin{array}{r} R2 \div 1 \to R2 \\ (-1)R2 + R1 \to R1 \\ (1/3)R2 + R3 \to R3 \\ (-1/3)R2 + R4 \to R4 \\ R2 + R5 \to R5 \end{array} \left[\begin{array}{cccccccc|c} 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 1/3 & 0 & -1/30 & 0 & 140/3 \\ 0 & 1 & 0 & 0 & -1/3 & 0 & -1/15 & 0 & 4/3 \\ 2 & 0 & 0 & 0 & 1 & 0 & 2/5 & 1 & -100 \end{array}\right]$$

The optimal solution is **maximum $w = -100$ at $x_1 = 0$, $x_2 = 4/3$, $x_3 = 140/3$**; therefore, the original problem has the optimal solution **$z = -w = 100$ at $x_1 = 0$, $x_2 = 4/3$, $x_3 = 140/3$**.